

RECOGNIZING $\mathrm{PSL}(2, p)$ IN THE NON-FRATTINI CHIEF FACTORS OF FINITE GROUPS

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ABSTRACT. Given a finite group G , let $P_G(s)$ be the probability that s randomly chosen elements generate G , and let H be a finite group with $P_G(s) = P_H(s)$. We show that if the nonabelian composition factors of G and H are $\mathrm{PSL}(2, p)$ for some non-Mersenne prime $p \geq 5$, then G and H have the same non-Frattini chief factors.

1. INTRODUCTION

Let G be a finite group. The probability $P_G(s)$ that s randomly chosen elements generate G is calculated as follows ([11]):

$$(1) \quad P_G(s) = \sum_{n \geq 1} \frac{a_n(G)}{n^s}, \text{ where } a_n(G) = \sum_{|G:H|=n} \mu_G(H).$$

Here μ_G is the Möbius function on the subgroup lattice of G defined recursively by $\mu_G(G) = 1$ and $\mu_G(H) = -\sum_{H < K \leq G} \mu_G(K)$ if $H < G$. Considering (1) as a formal Dirichlet series associated to G , if $G = \mathbb{Z}$ then

$$P_{\mathbb{Z}}(s) = \sum_{n \geq 1} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)},$$

where μ is the usual number-theoretic Möbius function and $\zeta(s)$ is the Riemann zeta function. The inverse of $P_G(s)$ is then called the *probabilistic zeta function* of G ; see [1] and [13].

Note that if $\mu_G(H) \neq 0$ then H is an intersection of maximal subgroups of G , cf. [11]. This implies $P_G(s) = P_{G/\mathrm{Frat}(G)}(s)$, where $\mathrm{Frat}(G)$ denotes the Frattini subgroup of G - the intersection of the maximal subgroups of G . Hence, one can only hope to get back information of $G/\mathrm{Frat}(G)$ from the knowledge of $P_G(s)$.

One natural question asks what we can say about G and H whenever $P_G(s) = P_H(s)$. It's known that if G is a simple group, then $H/\mathrm{Frat}(H) \cong G$, cf. [6, 16]. When G is not simple, the problem becomes much harder. Patassini makes a significant progress by obtaining the following results.

Theorem 1. [17] *Let G and H be finite groups with $P_G(s) = P_H(s)$. Then G and H have the same non-Frattini abelian chief factors.*

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Theorem 2. [18, Theorem 3] *Let G and H be finite groups whose nonabelian composition factors are alternating groups $\text{Alt}(k)$ where either $5 \leq k \leq 4.2 \cdot 10^{16}$ or $k \geq (e^{e^{15}} + 2)^3$. If $P_G(s) = P_H(s)$ then G and H have the same non-Frattini chief factors.*

Using the same method, we prove in this paper the following.

Theorem 3. *Let G and H be finite groups such that $P_G(s) = P_H(s)$. Assume that the nonabelian composition factors of G and H are $\text{PSL}(2, p)$, for some non-Mersenne prime $p \geq 5$. Then G and H have the same non-Frattini chief factors.*

From the proof of Theorem 3, one has the following consequence.

Corollary 4. *Let G and H be finite groups such that $P_G(s) = P_H(s)$. Assume that the nonabelian composition factors of G and H are either $\text{PSL}(2, p)$, for some non-Mersenne prime $p \geq 5$ or alternating groups $\text{Alt}(k)$ with k satisfying the hypothesis of Theorem 2. Then G and H have the same non-Frattini chief factors.*

In order to mimic the method from [18], we prove the following result.

Theorem 5. *Let L be a monolithic primitive group with socle $\text{soc}(L) \cong \text{PSL}(2, p)^n$ with $p \geq 5$ a non-Mersenne prime. Then the Dirichlet polynomial*

$$P_{L, \text{soc}(L)}(s) = \sum_{m \in \mathbb{N}} \frac{b_m(L, \text{soc}(L))}{m^s}, \text{ with } b_m(L, \text{soc}(L)) = \sum_{\substack{|L:H|=m \\ H \text{soc}(L)=L}} \mu_L(H)$$

is irreducible in the ring of finite Dirichlet polynomials.

Notations. In this paper, groups are always finite. Given a finite Dirichlet series $F(s) = \sum_{n \in \mathbb{N}} a_n/n^s$ and a set of prime numbers π , we denote by $F^{(\pi)}(s)$ the Dirichlet series obtained from $F(s)$ by deleting the a_n/n^s with n divisible by a prime in π . For a positive integer m , we denote by $\pi(m)$ the set of prime divisors of m . For a given finite group H , we write $\pi(H)$ for $\pi(|H|)$.

2. PRELIMINARIES

Given a normal subgroup N of G , it's shown in [2, Section 2.2] that

$$(2) \quad P_G(s) = P_{G/N}(s)P_{G,N}(s),$$

where

$$P_{G,N}(s) = \sum_{n \in \mathbb{N}} \frac{b_n(G, N)}{n^s}, \text{ with } b_n(G, N) = \sum_{\substack{|G:H|=n \\ HN=G}} \mu_G(H).$$

By taking a chief series

$$\Sigma : 1 = G_k < \cdots < G_1 < G_0 = G,$$

and iterating equation (2), we can express $P_G(s)$ as a product of Dirichlet polynomials indexed by the non-Frattini chief factors in Σ :

$$(3) \quad P_G(s) = \prod_{G_i/G_{i+1} \not\leq \text{Frat}(G/G_{i+1})} P_{G/G_{i+1}, G_i/G_{i+1}}(s).$$

It was proved in [8] that the factors in (3) are independent of the choice of the series Σ . Moreover, it also describes how those factors look like as follows.

Let A be a minimal normal subgroup of G . The monolithic primitive group associated to A is defined as

$$L_A := \begin{cases} A \rtimes G/C_G(A) & \text{if } A \text{ is abelian,} \\ G/C_G(A) & \text{otherwise.} \end{cases}$$

Note that $A \cong \text{soc}(L_A)$. Define

$$\begin{aligned} \tilde{P}_{L_A,1}(s) &= P_{L_A,A}(s), \\ \tilde{P}_{L_A,i}(s) &= P_{L_A,A}(s) - \frac{(1 + q_A + \cdots + q_A^{i-2})\gamma_A}{|A|^s}, \text{ for } i > 1, \end{aligned}$$

where $\gamma_A = |C_{\text{Aut}(A)(L_A/A)}|$ and $q_A = |\text{End}_{L_A}(A)|$ if A is abelian, $q_A = 1$ otherwise. If $A = H/K$ is a non-Frattini chief factor of G then $P_{G/K, H/K}(s) = \tilde{P}_{L_A,A}(s)$ where $\tilde{P}_{L_A,A}(s)$ is one of the $\tilde{P}_{L_A,i}(s)$ for a suitable choice i , cf. [8, Theorem 17].

If A is abelian then

$$P_{L_A,A}(s) = 1 - \frac{c(A)}{|A|^s},$$

where $c(A)$ is the number of complements of A in L_A , cf. [10]. Assume that $A \cong S_A^n$ is nonabelian. Let X_A be the subgroup of $\text{Aut}(S_A)$ induced by the conjugation action of the normalizer in L_A of a simple component of S_A^n . Then X_A is an almost simple group with socle S_A , cf. [5, Section 2].

Proposition 6. [12, Theorem 5]

$$\tilde{P}_{L_A,A}^{(r)}(s) = P_{L_A,A}^{(r)}(s) = P_{X_A, S_A}^{(r)}(ns - n + 1)$$

for every prime divisor r of the order of S_A .

3. PROOF OF THEOREM 5

In this section, we will prove Theorem 5. It enables us to obtain the irreducibility of the $\tilde{P}_{L_A,A}(s)$ in the factorization of $P_G(s)$. First of all, we will recall some useful results.

Let \mathcal{R} be the ring of Dirichlet polynomials with integer coefficients

$$\mathcal{R} = \left\{ \sum_{n \in \mathbb{N}} \frac{a_n}{n^s} : a_n \in \mathbb{Z}, \{n : a_n \neq 0\} | < \infty \right\}.$$

Given a set of prime numbers π , denote by X_π the set of commuting indeterminates $\{x_r : r \in \pi\}$. Let \mathcal{R}' be the subring of \mathcal{R} containing polynomials $\sum_{n \in \mathbb{N}} a_n/n^s$ such that $a_n \in n\mathbb{Z}$ for every n , and \mathcal{R}'_π its subring such that $a_n \neq 0$ whenever n is a π' number.

The rings $\mathcal{R}, \mathcal{R}'$ and \mathcal{R}'_π are factorial rings, cf.[7]. There is an isomorphism Φ between \mathcal{R}'_π and the polynomial ring $\mathbb{Z}[X_\pi]$ mapping p^{1-s} to x_p for every $p \in \pi$.

Lemma 7. [15, Lemma 10] *Let D be a factorial domain. If the polynomial $f(x) = 1 - ax^m \in D[x]$ is reducible then a or $-a$ is a non-trivial power in D .*

For a Dirichlet polynomial $F(s) = \sum_{n \in \mathbb{N}} a_n/n^s \in \mathcal{R}$ and v a prime number, denote by $|F(s)|_v$ the maximal v -part of n such that $a_n \neq 0$.

Lemma 8. [15, Lemma 12] *Let $h(s) = \sum_{k \in \mathbb{N}} a_k/s^k$ be a Dirichlet polynomial and let m be the least common multiple of $\{k : a_k \neq 0\}$. Assume the following hold:*

- *There exists a set of prime number π_0 such that $h^{(\pi_0)}(s)$ is irreducible.*
- *There exists a non-empty subset π of $\pi(m)$ such that $|h^{(\pi_0)}(s)|_v = |m|_v$ for all $v \in \pi$.*

Then $h(s)$ is irreducible in \mathcal{R} if and only if $(h(s), h^{(\pi)}(s)) = 1$.

Definition 9. Let $n \in \mathbb{N}_{>1}$. A prime number p is called a *primitive prime divisor* of $a^n - 1$ if it divides $a^n - 1$ but does not divide $a^e - 1$ for any integer $1 \leq e \leq n - 1$.

Proposition 10. [21] *Let a and n be integers greater than 1. There exists a primitive prime divisor of $a^n - 1$ except exactly in the following cases:*

- $n = 2, a = 2^s - 1$, where $s \geq 2$.
- $n = 6, a = 2$.

Notice that there may be more than one primitive prime divisor of $a^n - 1$. Such a prime is called a *Zsigmondy prime* for $\langle a, n \rangle$.

Proof of Theorem 5. If $p = 5$ then $\text{PSL}(2, 5) \cong \text{Alt}(5)$ and the result follows from [18, Theorem 2]. Assume now that $p > 5$ and set $S := \text{PSL}(2, p)$. Let t be the largest prime divisor of $(p - 1)$ such that t does not divide $(p + 1)$. Set $\pi_0 = \{t\}$. Since p is not a Mersenne prime, π_0 is non-empty. Let r be a Zsigmondy prime for $\langle p, 2 \rangle$ and denote by $x = x_r$ the indeterminate corresponding to the prime r . Let $D = \mathbb{Z}[X_{\pi(S) \setminus \{r\}}]$.

We first claim that $P_{L, \text{soc}(L)}^{(t)}(s)$ is irreducible. Let $X := X_S$ be its associated almost simple group as in Section 2. Note that $P_{L, \text{soc}(L)}^{(t)}(s) = P_{X, S}^{(t)}(ns - n + 1)$; cf. Proposition 6.

Case (i): $X = S = \text{PSL}(2, p)$. Then $P_{X, S}^{(t)}(ns - n + 1) = P_S^{(t)}(ns - n + 1)$. It follows from [14, Section 7] that

$$f(x) = \Phi(P_{L, \text{soc}(L)}^{(t)}(s)) = 1 - ax^m,$$

for some $m \in \mathbb{N}$ and $a = bx_p^n + c$ with $b, c \in \mathbb{Z}[X_{\pi(S) \setminus \{r, p\}}]$. By inspection, b and c are nonzero and hence a or $-a$ can not be a non-trivial power in D . Hence $P_{L, \text{soc}(L)}^{(t)}(s)$ is irreducible by Lemma 7.

Case (ii): $X = \text{PGL}(2, p)$. It follows from [4, Lemma 5] that

$$P_{X, S}(s) = - \sum_{H \leq S} \frac{\mu_X(H)}{|S : H|^s}.$$

The list of subgroups H and $\mu_X(H)$ are described in [4, Section 3]. By inspection and argument as in Case (i), we obtain the irreducibility of $P_{L,\text{soc}(L)}^{(t)}(s) = P_{X,S}^{(t)}(ns - n + 1)$.

Let $\pi = \{p, r\}$. One has that $|P_{L,\text{soc}(L)}^{(t)}(s)|_p = |\text{PSL}(2, p)|_p^n$ and $|P_{L,\text{soc}(L)}^{(t)}(s)|_r = |\text{PSL}(2, p)|_r^n$. Moreover $P_{L,\text{soc}(L)}^{(\pi \cup \{t\})}(s) = 1$. Hence it follows from Lemma 8 that $P_{L,\text{soc}(L)}(s)$ is irreducible. \square

Corollary 11. *Let L be a monolithic primitive group with socle $\text{soc}(L) \cong \text{PSL}(2, p)^n$, where $p \geq 5$ is a non-Mersenne prime. Then the Dirichlet polynomial $\tilde{P}_{L,\text{soc}(L)}(s)$ is irreducible.*

4. PROOF OF THEOREM 3

Analogous to [18, Proposition 28], we have the following crucial result.

Proposition 12. *Let X be an almost simple group with socle $\text{PSL}(2, p)$, and Z an almost simple group such that $P_{X,\text{PSL}(2,p)}^{(r)} = P_{Z,\text{soc}(Z)}^{(r)}(s)$ for every prime $r \leq p$. Then $\text{soc}(Z) \cong \text{PSL}(2, p)$.*

Proof. Let n be the minimal index of a proper subgroup of X which supplements $\text{PSL}(2, p)$.

If $p = 5$ then $n = 5$ and $\text{PSL}(2, 5) \cong \text{Alt}(5)$. The result follows from [18, Proposition 28]. If $p = 11$ then $X = \text{PSL}(2, 11)$, $n = 11$ and (cf. [14, Section 7])

$$P_{\text{PSL}(2,11)}(s) = 1 - \frac{22}{11^s} - \frac{12}{12^s} + \frac{66}{66^s} + \frac{220}{110^s} + \frac{132}{132^s} + \frac{165}{165^s} - \frac{220}{220^s} - \frac{990}{330^s} + \frac{660}{660^s}.$$

By GAP [9], the possibility for Z is that $Z = M_{11}$. However by considering $P_{\text{PSL}(2,11)}^{(2)}(s) = P_{Z,\text{soc}(Z)}^{(2)}(s)$ and noting that M_{11} has a maximal subgroup of index 55 (cf. [3]), we obtain a contradiction. This implies $\text{soc}(Z) \cong \text{PSL}(2, 11)$.

Assume now that $p > 11$. Then $n = p + 1$. Since $P_{X,\text{PSL}(2,p)}^{(p)}(s) = P_{Z,\text{soc}(Z)}^{(p)}(s)$ and $a_{p+1}(X, \text{PSL}(2, p)) \neq 0$; see [14, Section 7] and [4, Section 4], it follows that $a_{p+1}(X, \text{PSL}(2, p)) = a_{p+1}(Z, \text{soc}(Z)) \neq 0$. Thus the minimal index k of a proper subgroup of Z is at most $p + 1$. For a contradiction, assume that $k < p + 1$. Since $P_{X,\text{PSL}(2,p)}^{(p)}(s) = P_{Z,\text{soc}(Z)}^{(p)}(s)$, k should be divisible by k , so $k = p$. By considering $P_{X,\text{PSL}(2,p)}^{(2)}(s) = P_{Z,\text{soc}(Z)}^{(2)}(s)$, since $a_p(X, \text{PSL}(2, p)) = 0$ (see [14, Section 7] and [4, Section 4]), we get $a_p(Z, \text{soc}(Z)) = a_p(X, \text{PSL}(2, p)) = 0$, a contradiction. Hence $k = p + 1$. Since Z is almost simple, it is a primitive group of degree $p + 1$.

Since $P_{X,\text{PSL}(2,p)}^{(2)}(s) = P_{Z,\text{soc}(Z)}^{(2)}(s)$, [14, Section 7] and [4, Section 4] implies that p divides $|Z|$. Thus Z contains a p -cycle, since Z is a primitive group of degree $p + 1$ and $p > (p + 1)/2$. It follows from [20] that either $Z = M_{24}$ or $\text{soc}(Z) = \text{Alt}(p + 1)$ or $\text{soc}(Z) = \text{PSL}(2, p)$. If $Z = M_{24}$, then by considering $P_{X,\text{PSL}(2,23)}^{(2)}(s) = P_{Z,\text{soc}(Z)}^{(2)}(s)$ and noting that M_{24} has a maximal subgroup of index $7 \cdot 11 \cdot 23$, one gets a contradiction. Assume that $\text{soc}(Z) = \text{Alt}(p + 1)$. By Bertrand's postulate (cf. [19]), there exists a prime

l such that $(p+1)/2 < l < p-1$. By considering $P_{X, \text{PSL}(2,p)}^{(p)}(s) = P_{Z, \text{soc}(Z)}^{(p)}(s)$ one obtains from [5, Theorem 1.1] that l divides $|X|$, which is a contradiction. As a conclusion, we have that $\text{soc}(Z) \cong \text{PSL}(2, p)$. \square

Proof of Theorem 3. The proof is analogous to that of [18, Theorem 3]. We present here for the sake of completeness of the paper.

By Theorem 1, G and H have the same non-Frattini abelian chief factors. Thus we may assume that G and H have no non-Frattini abelian chief factors.

Let $\mathcal{CF}(G)$ be the set of the non-Frattini chief factors of G . For each $A \in \mathcal{CF}(G)$, the polynomial $\tilde{P}_{L,A}(s)$ is irreducible; cf. Corollary 11. Hence

$$P_G(s) = \prod_{A \in \mathcal{CF}(G)} \tilde{P}_{L,A}(s)$$

is a factorization of $P_G(s)$ into irreducible factors. Thus, there is a bijection between the sets $\mathcal{CF}(G)$ and $\mathcal{CF}(H)$ such that $A \cong S_A^{n_A} \in \mathcal{CF}(G)$ and $B \cong S_B^{n_B} \in \mathcal{CF}(H)$ are associated if and only if $\tilde{P}_{L,A}(s) = \tilde{P}_{L,B}(s)$. Since $\tilde{P}_{L,A}^{(r)}(s) = \tilde{P}_{L,B}^{(r)}(s)$ for every $r \in \pi(A)$. It follows that $n_A = n_B$, cf. [18, Proposition 27]. Thus $\tilde{P}_{X_A, S_A}^{(r)}(s) = \tilde{P}_{X_B, S_B}^{(r)}(s)$ for every $r \in \pi(A)$, cf. Proposition 6. Proposition 12 implies that $S_A \cong S_B$. Therefore $A \cong S_A^{n_A} \cong S_B^{n_B} \cong B$ as desired. \square

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